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Quantum aspects of Seiberg-Witten map in noncommutative Chern-Simons theory

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Abstract

Noncommutative Chern-Simons theory can be classically mapped to commutative Chern-Simons theory by the Seiberg-Witten map. We provide evidence that the equivalence persists at the quantum level by computing two and three-point functions of field strengths on the commutative side and their Seiberg-Witten transforms on the noncommutative side to the first nontrivial order in perturbation theory.

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1 Introduction

The Seiberg-Witten limit [1] is an interesting limit of open string theory with a constant NS-NS B field, in which open string dynamics reduces to a gauge theory defined on a noncommutative space. The theory in the limit can also be described in terms of fields defined on a commutative space. It was shown in [1] that these two descriptions are related to each other by a field redefinition called the Seiberg-Witten map. When the gauge group is $U(1)$, the Seiberg-Witten map has been obtained explicitly in [2, 3, 4] by studying the coupling of D-branes to the Ramond-Ramond potentials and by evaluating it in the Seiberg-Witten limit. It was shown that the field strength on the commutative space is expressed in terms of the open Wilson line [5, 6, 7] with appropriate insertions of operators on the noncommutative space.

Although the two descriptions are equivalent, fields on a noncommutative space are often more convenient in studying theories in the Seiberg-Witten limit. Actions expressed in terms of fields on a commutative space typically become nonpolynomial in the limit [1] and their closed forms are not known in general, though some constraints on possible terms in such actions have been studied in cases when they are realized as limits of string theory [8, 9, 10]. The lack of our understanding of actions on commutative spaces has prevented us from studying whether the equivalence implied by the Seiberg-Witten map holds at the quantum level.

One interesting case in which gauge theory actions are known in both descriptions is Chern-Simons theory in three dimensions. Its action in the noncommutative space is given by

$$S_{NCCS} = \frac{1}{2} \int d^3x \; \epsilon^{\mu\rho\nu} \mathrm{tr} \left[A_\mu * \partial_\rho A_\nu - \frac{2ig}{3} A_\mu * A_\rho * A_\nu \right], \tag{1.1}$$

where the product is the standard star-product:

$$f(x) * g(x) = \exp \left(\frac{i\theta^{\mu\nu}}{2} \frac{\partial}{\partial \xi^\mu} \frac{\partial}{\partial \zeta^\nu} \right) f(x+\xi) g(x+\zeta) \Big|_{\xi=\zeta=0}, \tag{1.2}$$

and tr is over the gauge group indices. It was shown [11, 12] that this action, when expressed in terms of gauge field $a_\mu(x)$ on a commutative space via the Seiberg-Witten map, becomes the standard Chern-Simons action. The proof of this statement is based on the differential equation characterizing the Seiberg-Witten map and holds for any $U(N)$ gauge group.

This is an interesting case for various reasons. First of all, since the actions are known in both descriptions and they both appear renormalizable, we can discuss the question of whether the equivalence of the two descriptions at the classical level can be extended to the quantum level. In this regard, there is an interesting puzzle. When the gauge group is $U(1)$, the Chern-Simons theory on the commutative space is trivial while its noncommutative counterpart has a cubic interaction. The latter theory seems to depend nontrivially on the coupling constant g, while the corresponding parameter for the former can be rescaled away. This casts some doubt on the quantum equivalence of the two. One of the motivations of this paper is to understand whether the equivalence in fact breaks down at the quantum level. We will compute correlation functions of open Wilson lines on the noncommutative space to the first nontrivial order in perturbation theory[4] and find that the equivalence persists at the quantum level.

The $U(1)$ Chern-Simons theory on the noncommutative space is expected to describe aspects of fractional quantum Hall fluid [15, 16, 17], and correlation functions of the open Wilson lines we will discuss in this paper play important roles in this context. Moreover it is known that such a theory is realized in a certain configuration of D-branes in string theory [18, 19]. We hope that the results in this paper shed some light on these issues.

When the gauge group is $U(N)$, an explicit form of the Seiberg-Witten map has not been derived in the sense of the works [2, 3, 4]. However, the map between a certain subset of observables on commutative and noncommutative sides can be extended to the $U(N)$ case. The generalization of our computations is straightforward, and we find that the equivalence holds for the $U(N)$ case as well.

The organization of the paper is as follows. In Section 2 we review the derivation of an exact expression for the Seiberg-Witten map and introduce a regularization for the composite operators appearing in the expression. We then calculate two and three-point functions of field strengths on the commutative side and their Seiberg-Witten transforms on the noncommutative side in perturbation theory in Section 3. We present our conclusions in Section 4, wherein we also discuss our generalization to the $U(N)$ case. Our conventions and Feynman rules are summarized in Appendix A, and some details of the computations in Section 3 are given in Appendix B.

2 Seiberg-Witten map and its regularization

2.1 Exact expression for the Seiberg-Witten map

An exact expression for the Seiberg-Witten map in arbitrary dimensions was derived by studying the Ramond-Ramond couplings of noncommutative gauge theory [2, 3, 4].[5] It takes a simple form in three dimensions. If we choose a coordinate system such that

[4]Perturbative aspects of noncommutative Chern-Simons theory have been studied in [13, 14].

[5]The Seiberg-Witten map from noncommuting to commuting variables is related to the Lagrange to Euler map in fluid mechanics [20, 15].

only θ^{12} and θ^{21} are nonvanishing, the Seiberg-Witten map is given by

$$
\begin{aligned}
f_{12}(k) &= -\frac{1}{g\theta^{12}}\left[W(k)-(2\pi)^3\delta^{(3)}(k)\right], \\
f_{0i}(k) &= O_{0i}(k) \quad \text{for} \quad i=1,2,
\end{aligned}
\tag{2.1}
$$

where $f_{\mu\nu}(k)$ is the field strength on the commutative side in momentum space, $W(k)$ is an open Wilson line

$$
W(k)=\int d^3x\ P_* \exp\left[ig\int_0^1 d\sigma\ l^\mu A_\mu(x+l\sigma)\right] * e^{ikx} \tag{2.2}
$$

with

$$
l^\mu \equiv (k\theta)^\mu = k_\nu\theta^{\nu\mu}, \tag{2.3}
$$

and $O_{\mu\nu}(k)$ is defined by

$$
O_{\mu\nu}(k)=\int d^3x\ P_* \exp\left[ig\int_0^1 d\sigma\ l^\mu A_\mu(x+l\sigma)\right] * F_{\mu\nu}(x) * e^{ikx} \tag{2.4}
$$

with

$$
F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu - igA_\mu * A_\nu + igA_\nu * A_\mu. \tag{2.5}
$$

Our convention for the path-ordered exponential is as follows:

$$
\begin{aligned}
&P_* \exp\left[ig\int_0^1 d\sigma\ l^\mu A_\mu(x+l\sigma)\right] \\
&= 1+ig\int_0^1 d\sigma_1\ l\cdot A(x+l\sigma_1) \\
&\quad + (ig)^2\int_0^1 d\sigma_1\int_0^{\sigma_1} d\sigma_2\ l\cdot A(x+l\sigma_1) * l\cdot A(x+l\sigma_2) + O(g^3).
\end{aligned}
\tag{2.6}
$$

The Seiberg-Witten map can also be written in the covariant form $f_{\mu\nu}(k)=O_{\mu\nu}(k)$ for $\mu,\nu=0,1,2$, as originally conjectured in [21]. However, the expression (2.1) is more convenient for our perturbative computations.

In [2] an expression of $f_{\mu\nu}(k;A_\mu,\theta)$ for arbitrary dimensions was constructed which (a) is gauge invariant,

$$
f_{\mu\nu}(k;A_\mu+\partial_\mu\lambda - igA_\mu * \lambda + ig\lambda * A_\mu,\theta) = f_{\mu\nu}(k;A_\mu,\theta), \tag{2.7}
$$

(b) obeys the Bianchi identity for the ordinary gauge theory:

$$
k_\mu f_{\rho\nu}(k)+k_\rho f_{\nu\mu}(k)+k_\nu f_{\mu\rho}(k)=0, \tag{2.8}
$$

(c) and satisfies the initial condition,

$$
\lim_{\theta\to 0} f_{\mu\nu}(k;A_\mu,\theta)=\int d^3x\ \left[\partial_\mu A_\nu(x)-\partial_\nu A_\mu(x)\right]e^{ikx}. \tag{2.9}
$$

A proof for arbitrary dimensions was given in [2], but it is much easier to see that $f_{\mu\nu}(k)$ in three dimensions defined by (2.1) satisfies these three conditions. The gauge invariance is guaranteed by the relation (2.3) [5], and the initial condition is easily verified. It is instructive to verify that the Seiberg-Witten map (2.1) satisfies the Bianchi identity $k_0 f_{12}(k) + k_1 f_{20}(k) + k_2 f_{01}(k) = 0$. Since $k_0 \delta^{(3)}(k) = 0$, what we need to show is $k_0 W(k) + g(k\theta)^\mu O_{0\mu}(k) = 0$. This can be shown as follows:

$$
\begin{aligned}
k_0 W(k) &= -i \int d^3x \; P_* \exp\left[ig \int_0^1 d\sigma \; l \cdot A(x + l\sigma)\right] * \partial_0 (e^{ikx}) \\
&= -g \int d^3x \; P_* \left[\exp\left[ig \int_0^1 d\sigma \; l \cdot A(x + l\sigma)\right] \int_0^1 d\sigma' \; l^\mu \partial_0 A_\mu (x + l\sigma')\right] * e^{ikx} \\
&= -g \int d^3x \; P_* \left[\exp\left[ig \int_0^1 d\sigma \; l \cdot A(x + l\sigma)\right] \right. \\
&\qquad\qquad \left. \times \int_0^1 d\sigma' \; l^\mu \{F_{0\mu}(x + l\sigma') + D_\mu A_0 (x + l\sigma')\}\right] * e^{ikx} \\
&= -g \int d^3x \; P_* \exp\left[ig \int_0^1 d\sigma \; l \cdot A(x + l\sigma)\right] * l^\mu F_{0\mu}(x) * e^{ikx} \\
&= -g(k\theta)^\mu O_{0\mu}(k),
\end{aligned}
\tag{2.10}
$$

where $D_\mu A_0 = \partial_\mu A_0 - igA_\mu * A_0 + igA_0 * A_\mu$. We integrated by parts in the first step, and then used the following identities:

$$
\int d^3x \; P_* \left[\exp\left[ig \int_0^1 d\sigma \; l \cdot A(x + l\sigma)\right] \int_0^1 d\sigma' \; l^\mu D_\mu A_0 (x + l\sigma')\}\right] * e^{ikx} = 0, \tag{2.11}
$$

which was shown in the appendix of [22] for the conservation of the energy-momentum tensor derived in the paper, and

$$
\begin{aligned}
& \int d^3x \; P_* \left[\exp\left[ig \int_0^1 d\sigma \; l \cdot A(x + l\sigma)\right] \mathcal{O}(x + l\sigma')\right] * e^{ikx} \\
= & \int d^3x \; P_* \exp\left[ig \int_0^1 d\sigma \; l \cdot A(x + l\sigma)\right] * \mathcal{O}(x) * e^{ikx}
\end{aligned}
\tag{2.12}
$$

for $0 \leq \sigma' \leq 1$, which is one of the basic properties of a straight open Wilson line [7].

It is well-known that the Seiberg-Witten map is not unique. However, the ambiguity pointed out in [23] is absent when the dimension of noncommutative directions is two, and the definition of the noncommutative gauge field is essentially unique in the Seiberg-Witten limit. The definition of the commutative gauge field may in general admit some ambiguity even in the Seiberg-Witten limit, but we assume that the expression (2.1) provides the map between noncommutative Chern-Simons theory and commutative Chern-Simons theory.

2.2 Regularization

When we compute correlation functions of $W(k)$ and $O_{\mu\nu}(k)$, we need to regularize these composite operators. A pure open Wilson line $W(k)$ is expanded in g as follows:

$$
\begin{aligned}
W(k) &= \int d^3x\ e^{ikx} + ig \int d^3x \int_0^1 d\sigma\ l\cdot A(x+l\sigma) * e^{ikx} \\
&\quad + (ig)^2 \int d^3x \int_0^1 d\sigma_1 \int_0^{\sigma_1} d\sigma_2\ l\cdot A(x+l\sigma_1) * l\cdot A(x+l\sigma_2) * e^{ikx} + O(g^3) \\
&= (2\pi)^3\delta^{(3)}(k) + ig\ l\cdot A(k) \\
&\quad + \frac{(ig)^2}{2} \int d^3x \int_0^1 d\sigma\ l\cdot A(x+l\sigma) * l\cdot A(x) * e^{ikx} + O(g^3). \qquad (2.13)
\end{aligned}
$$

The expression of $W(k)$ up to this order is sufficient for our perturbative computations in the next section. We regularize the composite operator at $O(g^2)$ as follows:

$$
\frac{(ig)^2}{2} \int d^3x \int_\epsilon^{1-\epsilon} d\sigma\ l\cdot A(x+l\sigma) * l\cdot A(x) * e^{ikx}. \qquad (2.14)
$$

Note that in addition to the expected singularity arising when $\sigma \to 0$, a singularity also arises when $\sigma \to 1$ since

$$
\begin{aligned}
&\int d^3x\ l\cdot A(x+l\sigma) * l\cdot A(x) * e^{ikx} = \int d^3x\ l\cdot A(x) * e^{ikx} * l\cdot A(x+l\sigma) \\
&= \int d^3x\ l\cdot A(x) * l\cdot A(x+l\sigma-l) * e^{ikx}, \qquad (2.15)
\end{aligned}
$$

where we have used the basic identities

$$
\int d^3x\ f(x) * g(x) = \int d^3x\ g(x) * f(x), \qquad (2.16)
$$

for any functions $f(x)$ and $g(x)$ which decay at infinity, and

$$
e^{ikx} * f(x+l) = f(x) * e^{ikx}, \qquad (2.17)
$$

for any C^∞ function $f(x)$. This regularization is natural for the following reason. In [22] it was shown how a straight open Wilson line arises from the computation of disk amplitudes in string theory. The integral over σ comes from the integral over a position of an open string vertex operator along the boundary. Since point-splitting regularization on the world-sheet boundary produces noncommutative gauge theory in space-time [1], it is natural to use point-splitting regularization for the integral over σ as well.

The operator $O_{\mu\nu}(k)$ (2.4) can also be expanded in g as follows:

$$
O_{\mu\nu}(k) = \int d^3x\ \left[\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\right] * e^{ikx}
$$

$$
\begin{aligned}
&-ig\int d^3x\ \left[A_\mu(x)*A_\nu(x)-A_\nu(x)*A_\mu(x)\right]*e^{ikx}\\
&+ig\int d^3x\int_0^1 d\sigma\ l\cdot A(x+l\sigma)*\left[\partial_\mu A_\nu(x)-\partial_\nu A_\mu(x)\right]*e^{ikx}+O(g^2).
\end{aligned}
\tag{2.18}
$$

The integral over σ in the last line can be regularized by taking the integration range from ϵ to $1-\epsilon$ as before. The commutator term in the second line is regularized as

$$
-ig\int d^3x\ \left[A_\mu(x+l\epsilon)*A_\nu(x)-A_\nu(x+l\epsilon)*A_\mu(x)\right]*e^{ikx}. \tag{2.19}
$$

Note that only the difference in two arguments matters. For example,

$$
\int d^3x\ A_\mu(x)*A_\nu(x-l\epsilon)*e^{ikx}=\int d^3x'\ A_\mu(x'+l\epsilon)*A_\nu(x')*e^{ikx'} \tag{2.20}
$$

with $x'=x-l\epsilon$ because of the fact that $l\cdot k=k_\mu\theta^{\mu\nu}k_\nu=0$.

This regularization for the commutator term is natural for the following reason. As is well-known, the commutator terms in the field strength arise from surface terms of the path-ordered integrals over positions of open string vertex operators [1, 22]. Therefore, if we use point-splitting regularization for the integral over σ, commutator terms should also be regularized correspondingly. The relation between the commutator term and the surface term in (2.18) at $O(g)$ can be seen, for example, by looking at the Bianchi identity $k_0W(k)+gl^\mu O_{0\mu}(k)=0$. Since $W(k)$ does not depend on $A_0(x)$, the part of $l^\mu O_{\mu 0}(k)$ which depends on $A_0(x)$ must cancel by itself. This can be seen for the regularized $O_{\mu\nu}(k)$ at order g as follows:

$$
\begin{aligned}
&l^\mu\left[ig\int d^3x\int_\epsilon^{1-\epsilon} d\sigma\ l\cdot A(x+l\sigma)*\partial_\mu A_0(x)*e^{ikx}\right]\\
&=ig\int d^3x\int_\epsilon^{1-\epsilon} d\sigma\ l\cdot\partial A_0(x)*l\cdot A(x+l\sigma-l)*e^{ikx}\\
&=ig\int d^3x\int_\epsilon^{1-\epsilon} d\sigma\ l\cdot\partial A_0(x+l-l\sigma)*l\cdot A(x)*e^{ikx}\\
&=-ig\int d^3x\int_\epsilon^{1-\epsilon} d\sigma\ \frac{\partial}{\partial\sigma}A_0(x+l-l\sigma)*l\cdot A(x)*e^{ikx}\\
&=-ig\int d^3x\,\{A_0(x+l\epsilon)-A_0(x+l-l\epsilon)\}*l\cdot A(x)*e^{ikx}\\
&=-ig\int d^3x\,\{A_0(x+l\epsilon)*l\cdot A(x)-l\cdot A(x)*A_0(x-l\epsilon)\}*e^{ikx}\\
&=l^\mu\left[-ig\int d^3x\,\{A_0(x+l\epsilon)*A_\mu(x)-A_\mu(x+l\epsilon)*A_0(x)\}*e^{ikx}\right],
\end{aligned}
\tag{2.21}
$$

where we have used the identities (2.16) and (2.17), and the change of variables $x'=x+l\sigma-l$. It is not difficult to see that the remaining part of the Bianchi identity which is independent of $A_0(x)$ also holds for a finite ϵ up to the current order in g. We therefore conclude that our regularization for the commutator (2.19) is in accord with point-splitting regularization of the integral over σ.

To summarize, we use the following regularized operators in terms of the gauge field in momentum space to compute correlation functions:

$$
\begin{aligned}
W(k) &= (2\pi)^3\delta^{(3)}(k) + ig\; l\cdot A(k) \\
&\quad + \frac{(ig)^2}{2}\int_\epsilon^{1-\epsilon} d\sigma \int \frac{d^3p}{(2\pi)^3}\; e^{-ik\times p\sigma + \frac{i}{2}k\times p}\; l\cdot A(p)\; l\cdot A(k-p) + O(g^3), \\
O_{\mu\nu}(k) &= -ik_\mu A_\nu(k) + ik_\nu A_\mu(k) \\
&\quad -ig\int \frac{d^3p}{(2\pi)^3}\; e^{-ik\times p\epsilon + \frac{i}{2}k\times p}\;\left[A_\mu(p)A_\nu(k-p) - A_\nu(p)A_\mu(k-p)\right] \\
&\quad +ig\int_\epsilon^{1-\epsilon} d\sigma \int \frac{d^3p}{(2\pi)^3}\; e^{-ik\times p\sigma + \frac{i}{2}k\times p} \\
&\qquad\qquad \times\, l\cdot A(p)\left[-i(k-p)_\mu A_\nu(k-p) + i(k-p)_\nu A_\mu(k-p)\right] \\
&\quad + O(g^2),
\end{aligned}
\tag{2.22}
$$

where we have introduced the notation $k\times p \equiv k_\mu \theta^{\mu\nu} p_\nu$.

3 Computations of correlation functions

3.1 Correlation functions on the commutative side

Correlation functions of field strengths can be easily calculated on the commutative side, where the action is given by

$$
S_{CS} = \frac{1}{2}\int d^3x\; \epsilon^{\mu\rho\nu} a_\mu \partial_\rho a_\nu. \tag{3.1}
$$

Since the field strength $f_{\mu\nu}$ can be expressed as

$$
f_{\mu\nu}(x) = \epsilon_{\mu\nu\rho}\frac{\delta S_{CS}}{\delta a_\rho(x)}, \tag{3.2}
$$

correlation functions can be easily evaluated by using the Schwinger-Dyson equations.

Correlation functions containing only f_{12} vanish because

$$
\begin{aligned}
&\langle f_{12}(x_1) f_{12}(x_2)\ldots f_{12}(x_n)\rangle = \int \mathcal{D}a\; f_{12}(x_1)f_{12}(x_2)\ldots f_{12}(x_n)\; e^{iS_{CS}} \\
&= -i\int \mathcal{D}a\; f_{12}(x_2)\ldots f_{12}(x_n)\;\frac{\delta}{\delta a_0(x_1)} e^{iS_{CS}} \\
&= -i\int \mathcal{D}a\;\frac{\delta}{\delta a_0(x_1)}\left[f_{12}(x_2)\ldots f_{12}(x_n)\; e^{iS_{CS}}\right] = 0.
\end{aligned}
\tag{3.3}
$$

Figure 1: Vanishing contractions.

The two-point function of f_{12} and f_{0i} is nonvanishing and is given by

$$\begin{aligned}\langle f_{12}(x) f_{0i}(y)\rangle &= \int \mathcal{D}a \ f_{12}(x) f_{0i}(y) \ e^{iS_{CS}} \\ &= -i \int \mathcal{D}a \ f_{0i}(y) \ \frac{\delta}{\delta a_0(x)} e^{iS_{CS}} = i \int \mathcal{D}a \ \frac{\delta f_{0i}(y)}{\delta a_0(x)} e^{iS_{CS}} = i \frac{\partial}{\partial x^i} \delta^{(3)}(x-y) \end{aligned} \quad (3.4)$$

for $i = 1, 2$. In momentum space, it is given by

$$\langle f_{12}(k) f_{0i}(k')\rangle = (2\pi)^3 k_i \delta^{(3)}(k+k') \quad \text{for} \quad i = 1, 2. \quad (3.5)$$

We will calculate the corresponding gauge-invariant observables on the noncommutative side to see if these results are reproduced.

3.2 $\langle W(k) W(k')\rangle$

One-point functions of $W(k)$ or $O_{\mu\nu}(k)$ are rather trivial because the length of an open Wilson line is proportional to the momentum k, while momentum conservation enforces $k = 0$. We cannot completely exclude possible subtleties arising from possible short-distance singularities, but in this paper we would rather study two-point and three-point functions which are more interesting.

Let us begin with the two-point function $\langle W(k) W(k')\rangle$. Since $k' = -k$ from momentum conservation, all of the gauge fields A_μ in each of two open Wilson lines are contracted with the same vector $(k\theta)^\mu$ up to a sign. If we choose a coordinate system such that $k_2 = 0$ by a rotation in the (x^1, x^2)-plane, only A_2 appears in the open Wilson lines since the only nonvanishing component of $(k\theta)^\mu$ is $\mu = 2$. Therefore, the two-point function $\langle W(k) W(k')\rangle$ consists of correlation functions involving only A_2, $\langle A_2(p_1) A_2(p_2) \ldots A_2(p_n)\rangle$.

In noncommutative Chern-Simons theory, the propagator $\langle A_2(p) A_2(q)\rangle$ vanishes in the Landau gauge.[6] Therefore, we cannot contract any pair of gauge fields coming from the two Wilson lines directly. Contractions such as the ones shown in Figure 1 are prohibited. This rule also applies to $\langle W(k) O_{\mu\nu}(k')\rangle$ which we will discuss in the next subsection. From this it immediately follows that $O(g^2)$ contribution to $\langle W(k) W(k')\rangle$ vanishes:

$$\langle W(k) W(k')\rangle = (2\pi)^6 \delta^{(3)}(k) \delta^{(3)}(k') + O(g^4). \quad (3.6)$$

[6] Feynman rules are summarized in Appendix A.

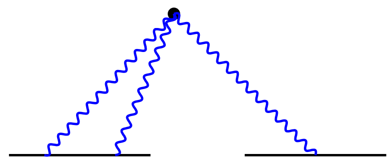

Figure 2: Vanishing interaction graph.

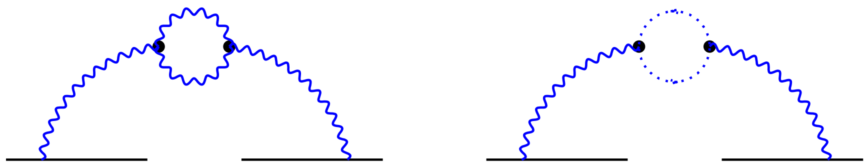

Figure 3: One-loop propagator correction insertions. The dotted line denotes the ghost propagator.

Furthermore, even if the cubic vertices are used, we end up with at least one contraction of $\langle A_2(p)A_2(q)\rangle$ unless there is at least one internal loop. Let us take the diagram shown in Figure 2 as an example. The cubic vertex of noncommutative Chern-Simons theory consists of a product of A_0, A_1, and A_2. Therefore, one of the three contractions in Figure 2 must be $\langle A_2A_2\rangle$.

Now consider diagrams with internal loops. To lowest nontrivial order, $O(g^4)$, this corresponds to the diagrams in Figure 3.[7] These involve the one-loop corrections to the gauge field propagator. The calculation of these corrections is similar to those in commutative *non-Abelian* Chern-Simons theory [14]. Both diagrams generate the same noncommutative phase structure, and both can be broken respectively into planar and nonplanar parts in the standard way [24, 25]. The nonplanar pieces are regulated by the noncommutative phases [26, 24, 25],[8] after which the contributions from the ghost loop and gauge loop rigorously cancel. On the other hand, the planar pieces of these diagrams, which are identical to their commutative counterparts (up to the same overall factor), require careful regularization, and the study of which ultimately yields the famous one-loop shift to the Chern-Simons coupling [27, 28, 29]. However, the one-loop corrections to the Chern-Simons propagator itself change neither its tensor structure, nor its momentum dependence. Thus the arguments of the previous paragraph apply: the one-loop corrections to $\langle A_2(p)A_2(q)\rangle$ still vanish, and so we conclude that $O(g^4)$ contribution to $\langle W(k)W(k')\rangle$ also vanishes:

$$\langle W(k)W(k')\rangle = (2\pi)^6\delta^{(3)}(k)\delta^{(3)}(k') + O(g^6). \tag{3.7}$$

This is consistent with the equivalence between noncommutative Chern-Simons theory

[7] The second diagram involves the ghost loop arising from the usual gauge fixing of the theory, which we have not presented, and which we do not require in the sequel.

[8] See also the paragraph containing (3.15) in the next subsection.

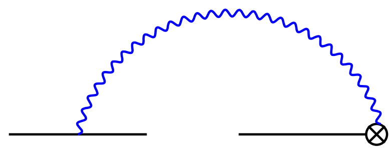

Figure 4: Lowest-order contribution to $\langle W(k) O_{\mu\nu}(k')\rangle$. The cross on the right open Wilson line denotes the field strength insertion.

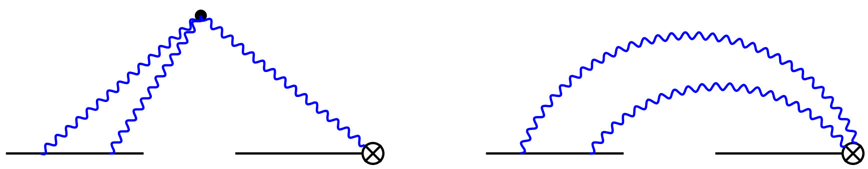

Figure 5: Diagrams 1 and 2. Canceling $O(g^3)$ corrections to $\langle W(k) O_{\mu\nu}(k')\rangle$ involving two gauge field sources on the pure open Wilson line.

and commutative Chern-Simons theory.

3.3 $\langle W(k) O_{\mu\nu}(k')\rangle$

The lowest-order term in $\langle W(k) O_{\mu\nu}(k')\rangle$ reproduces the result from the commutative side (3.5) by construction and corresponds to the diagram in Figure 4. Let us verify this explicitly.

$$\begin{aligned}
\langle W(k) O_{\mu\nu}(k')\rangle &= ig(k\theta)^\rho(-ik'_\mu)\,\langle A_\rho(k) A_\nu(k')\rangle - (\mu \leftrightarrow \nu) + O(g^3) \\
&= -g(2\pi)^3\delta^{(3)}(k+k')(k\theta)^\rho k^\sigma \frac{k_\mu \epsilon_{\rho\sigma\nu} - k_\nu \epsilon_{\rho\sigma\mu}}{k^2} + O(g^3). \qquad (3.8)
\end{aligned}$$

Since the term proportional to $\delta^{(3)}(k)$ in (2.1) is not relevant to the current calculation, the result (3.5) should be reproduced by $\langle W(k) O_{0i}(k')\rangle$ divided by $-g\theta^{12}$. In fact,

$$-\frac{1}{g\theta^{12}}\,\langle W(k) O_{0i}(k')\rangle = (2\pi)^3 k_i \delta^{(3)}(k+k') + O(g^2) \quad \text{for} \quad i = 1,2. \qquad (3.9)$$

Therefore, the question is whether higher-order terms modify this result or not. We will calculate $\langle W(k) O_{\mu\nu}(k')\rangle$ at $O(g^3)$.

Feynman diagrams at $O(g^3)$ fall into two categories. The first one contains diagrams which do not have an internal loop. There are five diagrams in this category, shown in Figures 5 and 6. The second one contains diagrams involving the one-loop correction to the propagator, which are displayed in Figure 7.

Calculations of the five diagrams in the first category are given in Appendix B. Here we only present the final results.

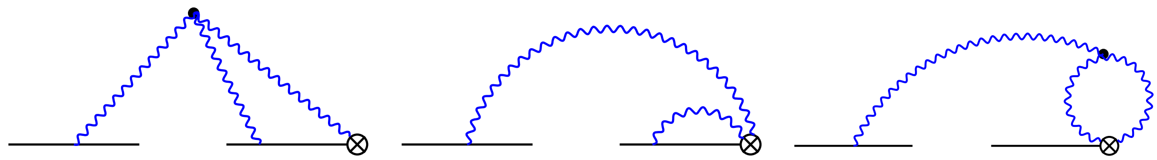

Figure 6: Diagrams 3, 4 and 5. Canceling $O(g^3)$ corrections to $\langle W(k)O_{\mu\nu}(k')\rangle$ involving one gauge field source on the pure open Wilson line.

Diagram 1

$$\begin{aligned}
&\langle W(k)O_{\mu\nu}(k')\rangle_{\text{Diagram 1}} \\
&= -(ig)^3\delta^{(3)}(k+k')\int_{\epsilon}^{1-\epsilon} d\sigma \int d^3p\ \frac{e^{-ik\times p\sigma}}{p^2(k-p)^2} \\
&\qquad \times\Big\{(k\times p)(k_\mu l_\nu - k_\nu l_\mu) - l^2(k_\mu p_\nu - k_\nu p_\mu)\Big\}. \qquad (3.10)
\end{aligned}$$

Diagram 2

$$\begin{aligned}
&\langle W(k)O_{\mu\nu}(k')\rangle_{\text{Diagram 2}} \\
&= (ig)^3\delta^{(3)}(k+k')\int_{0}^{1-2\epsilon} d\sigma \int d^3p\ \frac{e^{-ik\times p\sigma}}{p^2(k-p)^2} \\
&\qquad \times\Big\{(k\times p)(k_\mu l_\nu - k_\nu l_\mu) - l^2(k_\mu p_\nu - k_\nu p_\mu)\Big\}. \qquad (3.11)
\end{aligned}$$

Diagram 3

$$\begin{aligned}
&\langle W(k)O_{\mu\nu}(k')\rangle_{\text{Diagram 3}} \\
&= 2(ig)^3\delta^{(3)}(k+k')\int_{\epsilon}^{1-\epsilon} d\sigma \int d^3p\ e^{ik\times p\sigma}\ \frac{l^2(k_\mu p_\nu - k_\nu p_\mu)}{k^2p^2} \\
&\quad -i(ig)^3\delta^{(3)}(k+k')\int d^3p\ \frac{e^{ik\times p(1-\epsilon)} - e^{ik\times p\epsilon}}{k^2p^2(k+p)^2} \\
&\qquad \times\Big[-(k^2+k\cdot p)(k_\mu l_\nu - k_\nu l_\mu) - (k^2+2k\cdot p)(p_\mu l_\nu - p_\nu l_\mu) \\
&\qquad\qquad +2(k\times p)(p_\mu k_\nu - p_\nu k_\mu)\Big]. \qquad (3.12)
\end{aligned}$$

Diagram 4

$$\begin{aligned}
&\langle W(k)O_{\mu\nu}(k')\rangle_{\text{Diagram 4}} \\
&= -2(ig)^3\delta^{(3)}(k+k')\int_{\epsilon}^{1-2\epsilon} d\sigma \int d^3p\ e^{ik\times p\sigma}\ \frac{l^2(k_\mu p_\nu - k_\nu p_\mu)}{k^2p^2} \\
&\quad -2i(ig)^3\delta^{(3)}(k+k')\int d^3p\ \Big\{e^{ik\times p(1-2\epsilon)} - e^{ik\times p\epsilon}\Big\}\frac{k_\mu l_\nu - k_\nu l_\mu}{k^2p^2}. \qquad (3.13)
\end{aligned}$$

Diagram 5

$$
\begin{aligned}
&\langle W(k)O_{\mu\nu}(k')\rangle_{\text{Diagram 5}} \\
&= i(ig)^3\delta^{(3)}(k+k')\int d^3p\ \frac{e^{ik\times p(1-\epsilon)}-e^{ik\times p\epsilon}}{k^2p^2(k+p)^2} \\
&\quad\times\Big[-2(k\times p)(k_\mu p_\nu-k_\nu p_\mu)-(2p^2+k\cdot p)(l_\mu k_\nu-l_\nu k_\mu) \\
&\qquad +(k^2+2k\cdot p)(l_\mu p_\nu-l_\nu p_\mu)\Big].
\end{aligned} \tag{3.14}
$$

Let us first consider whether or not each of the five contributions is finite. All of the integrals over the momentum p take the following form:

$$
\int d^3p\ f(p_0,p_1,p_2)\ e^{ik\times p\sigma}, \tag{3.15}
$$

where $f(p_0,p_1,p_2)$ is a meromorphic function of p_0, p_1, and p_2. If the θ-dependent phase factor is absent, the integral can be divergent. However, as is well-known [24, 25], the phase factor makes the integral convergent. Let us illustrate this point in the simple example where $f(p_0,p_1,p_2)=1/[(2\pi)^3(p_0^2+p_1^2+p_2^2)]$. We can choose a coordinate system such that $k_2=0$. The integral becomes

$$
\int\frac{d^3p}{(2\pi)^3}\ \frac{e^{ik_1\theta^{12}p_2\sigma}}{p_0^2+p_1^2+p_2^2}. \tag{3.16}
$$

The integral over p_2 can be carried out by evaluating the residue of the pole at either $p_2=i\sqrt{p_0^2+p_1^2}$ or $p_2=-i\sqrt{p_0^2+p_1^2}$ depending on the sign of $k_1\theta^{12}\sigma$. The remaining integrals over p_0 and p_1 are easily performed to give

$$
\int_{-\infty}^{\infty}\frac{dp_0}{2\pi}\int_{-\infty}^{\infty}\frac{dp_1}{2\pi}\ \frac{e^{-|k_1\theta^{12}\sigma|\sqrt{p_0^2+p_1^2}}}{2\sqrt{p_0^2+p_1^2}}=\frac{1}{4\pi|k_1\theta^{12}\sigma|}. \tag{3.17}
$$

This is nothing but the calculation of Green's function in three dimensions if we replace $(k\theta)^\mu\sigma$ by x^μ. Note that the integral over p_2 provided a damping factor which exponentially suppresses the integrand for large p and thus makes the integrals over p_0 and p_1 converge. This mechanism works in general as long as the phase factor is nonvanishing.

Therefore, the integrals over p in the five diagrams converge as long as σ is nonzero. Only the integral in Diagram 2 is potentially dangerous, but we can show that it is convergent as well. What we need to show is the following:

$$
\lim_{\epsilon\to 0}\int_0^\epsilon d\sigma\int d^3p\ \frac{e^{-ik\times p\sigma}}{p^2(k-p)^2}\Big\{(k\times p)(k_\mu l_\nu-k_\nu l_\mu)-l^2(k_\mu p_\nu-k_\nu p_\mu)\Big\}=0. \tag{3.18}
$$

For the first term, only the surface terms of the integral over σ contribute:

$$i(k_\mu l_\nu - k_\nu l_\mu) \int d^3p \; \frac{e^{-ik\times p\epsilon} - 1}{p^2(k-p)^2}. \tag{3.19}$$

Now the term coming from $\sigma = 0$ is also finite by power counting so that we can safely take the limit $\epsilon \to 0$. The calculation of the second term reduces to

$$\int d^3p \; \frac{p_\mu \; e^{-ik_1\theta^{12}p_2\sigma}}{(\frac{k}{2}-p)^2(\frac{k}{2}+p)^2}, \tag{3.20}$$

where we have changed variables as $p' = k/2 - p$ and chosen a coordinate system such that $k_2 = 0$ as usual. When $p_\mu = p_2$, the calculation reduces to the case of the first term in (3.18). When $p_\mu = p_0$ or $p_\mu = p_1$, the integral vanishes for a nonzero σ because the integrand after the integral over p_2 is odd in $(p_0, p_1) \to (-p_0, -p_1)$. Thus we have shown (3.18) and confirmed that each of the contributions coming from the five diagrams is finite.

The contributions from Diagram 1 and Diagram 2 almost cancel. The sum of the two can be written as follows:

$$\begin{aligned}(ig)^3\delta^{(3)}(k+k')\Bigg[&\int_0^\epsilon d\sigma \int d^3p \; \frac{e^{-ik\times p\sigma}}{p^2(k-p)^2}\Big\{(k\times p)(k_\mu l_\nu - k_\nu l_\mu) - l^2(k_\mu p_\nu - k_\nu p_\mu)\Big\} \\ &- \int_{1-2\epsilon}^{1-\epsilon} d\sigma \int d^3p \; \frac{e^{-ik\times p\sigma}}{p^2(k-p)^2}\Big\{(k\times p)(k_\mu l_\nu - k_\nu l_\mu) - l^2(k_\mu p_\nu - k_\nu p_\mu)\Big\}\Bigg]. \end{aligned} \tag{3.21}$$

As we have seen, the first term vanishes in the limit $\epsilon \to 0$. The second term is less dangerous and also vanishes in the limit. We thus conclude that the contributions from Diagram 1 and Diagram 2 cancel.

Now consider the remaining three diagrams. The contributions from Diagram 3 and Diagram 4 contain integrals over σ. The two integrals almost cancel and the difference vanishes in the limit $\epsilon \to 0$ as before. The remaining terms which do not contain an integral over σ share a similar structure. The term in Diagram 4 is different in that it contains $e^{ik\times p(1-2\epsilon)}$. However, we can replace it by $e^{ik\times p(1-\epsilon)}$ since the difference vanishes in the limit $\epsilon \to 0$:

$$\lim_{\epsilon\to 0} \int d^3p \; \Big\{e^{ik\times p(1-2\epsilon)} - e^{ik\times p(1-\epsilon)}\Big\} \frac{k_\mu l_\nu - k_\nu l_\mu}{k^2p^2} = 0. \tag{3.22}$$

Now the sum of the terms from the three diagrams can be written as follows:

$$\begin{aligned}&-i(ig)^3\delta^{(3)}(k+k') \int d^3p \; \frac{e^{ik\times p(1-\epsilon)} - e^{ik\times p\epsilon}}{k^2p^2(k+p)^2}(k^2 + 2k\cdot p)(k_\mu l_\nu - k_\nu l_\mu) \\ &= -i(ig)^3\delta^{(3)}(k+k') \int d^3p \; \frac{e^{ik\times p(1-\epsilon)} - e^{ik\times p\epsilon}}{k^2}\left[\frac{1}{p^2} - \frac{1}{(k+p)^2}\right](k_\mu l_\nu - k_\nu l_\mu) = 0,\end{aligned} \tag{3.23}$$

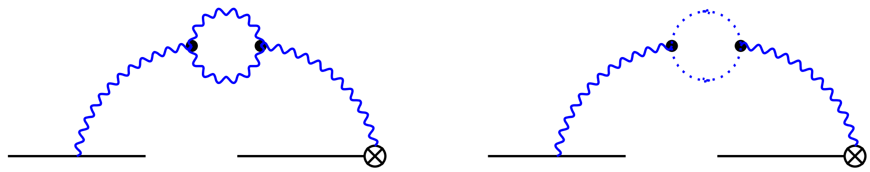

Figure 7: $O(g^3)$ contributions to $\langle W(k)O_{\mu\nu}(k')\rangle$ containing propagator corrections.

where we have changed variables as $p' = p + k$ for the term involving $1/(k+p)^2$ in the last line. To summarize, we have seen that in the limit $\epsilon \to 0$ the contributions from Diagram 1 and Diagram 2 cancel, and those from Diagram 3, 4, and 5 cancel among themselves.

The second category of diagrams at $O(g^3)$, displayed in Figure 7, contain the one-loop correction to the propagator. As we have discussed in the previous subsection, the nonplanar contributions from these diagrams are finite and cancel between the gauge-field and ghost diagrams, and the planar contributions only renormalize the overall coefficient of the tree-level result:

$$-\frac{1}{g\theta^{12}}\langle W(k)O_{0i}(k')\rangle = \left[1 + O(g^2)\right](2\pi)^3 k_i \delta^{(3)}(k+k') + O(g^4) \quad \text{for} \quad i = 1, 2. \tag{3.24}$$

Does this violate the equivalence between noncommutative and commutative Chern-Simons theories?

The disagreement is coming from different wave-function renormalizations between the commutative and noncommutative theories. The commutative $U(1)$ Chern-Simons theory is free and its propagator $\langle a_\mu(p)a_\nu(q)\rangle$ does not receive any wave-function renormalization. On the other hand, the tree-level propagator $\langle A_\mu(p)A_\nu(q)\rangle$ in the noncommutative theory can be renormalized by quantum effects depending on a regularization scheme, and this is precisely the origin of the one-loop correction to $\langle W(k)O_{\mu\nu}(k')\rangle$. However, since the correction changes neither the tensor structure nor the momentum dependence and only modifies the overall coefficient, its effect can be absorbed in renormalizations of the composite operators $W(k)$ and $O_{\mu\nu}(k)$. Therefore, the equivalence between the commutative and noncommutative theories still holds if we modify the Seiberg-Witten map at the classical level (2.1) to

$$\begin{aligned} f_{12}(k) &= -\frac{Z}{g\theta^{12}}\left[W(k) - (2\pi)^3\delta^{(3)}(k)\right], \\ f_{0i}(k) &= Z O_{0i}(k) \quad \text{for} \quad i = 1, 2, \end{aligned} \tag{3.25}$$

such that the renormalization factor Z compensates for the correction to the two-point function $\langle W(k)O_{\mu\nu}(k')\rangle$. Note that the two renormalization factors in (3.25) must be the same in order for the Bianchi identity to hold.

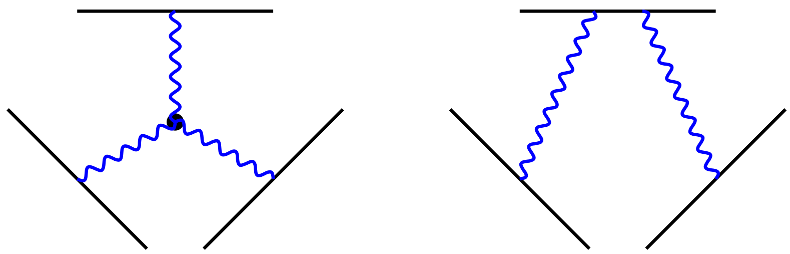

Figure 8: $O(g^4)$ contributions to $\langle W(k_1)W(k_2)W(k_3)\rangle$.

It is not too surprising that we found a scheme-dependent quantum correction to the Seiberg-Witten map because the Seiberg-Witten map is a special kind of field redefinition between the commutative variable a_μ and the noncommutative variable A_μ, and these fields in general suffer from scheme-dependent wave-function renormalizations. Our expression for the Seiberg-Witten map was designed to satisfy the initial condition (2.9) classically, but a quantum correction is necessary if the wave-function renormalization is singular in the limit $\theta \to 0$ as it is in the case of noncommutative Chern-Simons theory.

3.4 $\langle W(k_1)W(k_2)W(k_3)\rangle$

Let us calculate the three-point function of pure open Wilson lines. This should vanish except for the trivial lowest-order term in order for the correspondence between the commutative and noncommutative sides to hold.

The first nontrivial contribution starts at $O(g^4)$. Let us expand $\langle W(k_1)W(k_2)W(k_3)\rangle$ up to $O(g^4)$:

$$
\begin{aligned}
&\langle W(k_1)W(k_2)W(k_3)\rangle \\
&= (2\pi)^9\delta^{(3)}(k_1)\delta^{(3)}(k_2)\delta^{(3)}(k_3) \\
&\quad + (ig)^3 \langle l_1 \cdot A(k_1)\ l_2 \cdot A(k_2)\ l_3 \cdot A(k_3)\rangle \\
&\quad + \frac{(ig)^4}{2}\int_\epsilon^{1-\epsilon} d\sigma \int \frac{d^3p}{(2\pi)^3} \\
&\qquad \times \Big[e^{-ik_3\times p\sigma + \frac{i}{2}k_3\times p} \langle l_1 \cdot A(k_1)\ l_2 \cdot A(k_2)\ l_3 \cdot A(p)\ l_3 \cdot A(k_3-p)\rangle \\
&\qquad\qquad +((k_1,k_2,k_3)\to(k_2,k_3,k_1)) + ((k_1,k_2,k_3)\to(k_3,k_1,k_2))\Big] \\
&\quad + O(g^5),
\end{aligned}
\tag{3.26}
$$

where $l_i^\mu \equiv (k_i\theta)^\mu = (k_i)_\nu\theta^{\nu\mu}$ for $i = 1,2,3$. The term at $O(g^3)$ contracted with the vertex (A.4) and the terms at $O(g^4)$ contracted with two propagators of (A.3) contribute at $O(g^4)$. These correspond to the diagrams in Figure 8.

Let us begin with the latter. There are two nonvanishing contractions for each of the three terms at $O(g^4)$. The two contractions are combined to give the following expression:

$$
\begin{aligned}
&\frac{(ig)^4}{2}\int_\epsilon^{1-\epsilon} d\sigma \int \frac{d^3p}{(2\pi)^3}\; e^{-ik_3\times p\sigma+\frac{i}{2}k_3\times p} \\
&\qquad\qquad \times \langle l_1\cdot A(k_1)\; l_2\cdot A(k_2)\; l_3\cdot A(p)\; l_3\cdot A(k_3-p)\rangle \\
&= (ig)^4(2\pi)^3\delta^{(3)}(k_1+k_2+k_3)\int_\epsilon^{1-\epsilon} d\sigma\;\cos\left[k_1\times k_2\left(\sigma-\frac{1}{2}\right)\right] \\
&\qquad\qquad \times l_1^{\mu_1}l_2^{\mu_2}l_3^{\nu_1}l_3^{\nu_2}\epsilon_{\mu_1\rho_1\nu_1}\epsilon_{\mu_2\rho_2\nu_2}\frac{k_1^{\rho_1}}{k_1^2}\frac{k_2^{\rho_2}}{k_2^2}.
\end{aligned}
\tag{3.27}
$$

Since the vectors l_i^μ have vanishing 0-components, the indices μ_1, ν_1, μ_2, and ν_2 cannot be zero. Therefore, the indices ρ_1 and ρ_2 must be zero in order for the expression to be nonvanishing. If we decompose the vectors k_i as

$$
k_i = (\omega_i, \vec{k}_i)
\tag{3.28}
$$

for $i=1,2,3$ where $\omega_i = (k_i)^0$ and $(\vec{k}_i)^\mu = (k_i)^\mu$ for $\mu=1,2$, we have

$$
\begin{aligned}
&l_1^{\mu_1}l_2^{\mu_2}l_3^{\nu_1}l_3^{\nu_2}\epsilon_{\mu_1\rho_1\nu_1}\epsilon_{\mu_2\rho_2\nu_2}\frac{k_1^{\rho_1}}{k_1^2}\frac{k_2^{\rho_2}}{k_2^2} \\
&= \frac{(\theta^{12})^2\omega_1\omega_2(k_1\times k_3)(k_2\times k_3)}{k_1^2k_2^2} = -\frac{(\theta^{12})^2\omega_1\omega_2(k_1\times k_2)^2}{k_1^2k_2^2},
\end{aligned}
\tag{3.29}
$$

where we have used

$$
l_i^\mu\epsilon_{\mu 0\nu}l_j^\nu = -\theta^{12}(k_i\times k_j),
\tag{3.30}
$$

and $k_1\times k_2 = k_2\times k_3 = k_3\times k_1$ by momentum conservation. The integral over σ is straightforward and we do not have any singularity when we take $\epsilon\to 0$. The result is thus given by

$$
-2(ig)^4(2\pi)^3\delta^{(3)}(k_1+k_2+k_3)\sin\left(\frac{k_1\times k_2}{2}\right)\frac{(\theta^{12})^2k_3^2\omega_1\omega_2(k_1\times k_2)}{k_1^2k_2^2k_3^2}.
\tag{3.31}
$$

We also need to add the two terms coming from the permutations $(k_1,k_2,k_3)\to(k_2,k_3,k_1)$ and $(k_1,k_2,k_3)\to(k_3,k_1,k_2)$. Since $k_1^2k_2^2k_3^2$ is invariant under the permutations and $k_1\times k_2 = k_2\times k_3 = k_3\times k_1$, the only nontrivial part is $k_3^2\omega_1\omega_2$. We can eliminate ω_3 and $\vec{k}_3$ using momentum conservation to find

$$
\begin{aligned}
&k_3^2\omega_1\omega_2 + k_1^2\omega_2\omega_3 + k_2^2\omega_3\omega_1 \\
&= 2\omega_1\omega_2\vec{k}_1\cdot\vec{k}_2 - \omega_2^2\vec{k}_1^2 - \omega_1^2\vec{k}_2^2 = \frac{2\omega_1\omega_2 l_1\cdot l_2 - \omega_2^2 l_1^2 - \omega_1^2 l_2^2}{(\theta^{12})^2},
\end{aligned}
\tag{3.32}
$$

where we have used

$$\vec{k}_i \cdot \vec{k}_j = \frac{1}{(\theta^{12})^2} l_i \cdot l_j. \tag{3.33}$$

Therefore, the contribution to $\langle W(k_1)W(k_2)W(k_3)\rangle$ at $O(g^4)$ coming from the sum of the diagrams without cubic vertices is given by

$$2(ig)^4(2\pi)^3\delta^{(3)}(k_1+k_2+k_3)\sin\left(\frac{k_1\times k_2}{2}\right)\frac{(-2\omega_1\omega_2 l_1\cdot l_2+\omega_2^2 l_1^2+\omega_1^2 l_2^2)(k_1\times k_2)}{k_1^2k_2^2k_3^2}. \tag{3.34}$$

Let us next calculate the contribution from the diagram with a cubic vertex. It can be evaluated using (A.4) and (A.6) as follows:

$$\begin{aligned}
&(ig)^3\left\langle l_1\cdot A(k_1)\ l_2\cdot A(k_2)\ l_3\cdot A(k_3)\right\rangle \\
&= -2(ig)^4(2\pi)^3\delta^{(3)}(k_1+k_2+k_3)\sin\left(\frac{k_1\times k_2}{2}\right) \\
&\quad\times\frac{-2(k_1\times k_2)^3+(k_1\times k_2)\{k_1^2l_2^2+k_2^2l_1^2-2(k_1\cdot k_2)(l_1\cdot l_2)\}}{k_1^2k_2^2k_3^2}.
\end{aligned} \tag{3.35}$$

Apparently, this does not seem to cancel the contribution (3.34). For example, the expression (3.34) vanishes when $\omega_1=\omega_2=0$, but it is not obvious that this also holds in (3.35). Let us take a closer look at the factor $k_1^2l_2^2+k_2^2l_1^2-2(k_1\cdot k_2)(l_1\cdot l_2)$. It can be decomposed in the following way:

$$\begin{aligned}
&k_1^2l_2^2+k_2^2l_1^2-2(k_1\cdot k_2)(l_1\cdot l_2)=(\omega_1^2+\vec{k}_1^2)l_2^2+(\omega_2^2+\vec{k}_2^2)l_1^2-2(\omega_1\omega_2+\vec{k}_1\cdot\vec{k}_2)l_1\cdot l_2 \\
&=\omega_1^2l_2^2+\omega_2^2l_1^2-2\omega_1\omega_2 l_1\cdot l_2+2(\theta^{12})^2\{\vec{k}_1^2\vec{k}_2^2-(\vec{k}_1\cdot\vec{k}_2)^2\} \\
&=\omega_1^2l_2^2+\omega_2^2l_1^2-2\omega_1\omega_2 l_1\cdot l_2+2(k_1\times k_2)^2,
\end{aligned} \tag{3.36}$$

where we have used (3.33) and

$$(k_i\times k_j)^2=(\vec{k}_i\times\vec{k}_j)^2=(\theta^{12})^2\{\vec{k}_i^2\vec{k}_j^2-(\vec{k}_i\cdot\vec{k}_j)^2\}. \tag{3.37}$$

Therefore, we have

$$\begin{aligned}
&(ig)^3\left\langle l_1\cdot A(k_1)\ l_2\cdot A(k_2)\ l_3\cdot A(k_3)\right\rangle \\
&= -2(ig)^4(2\pi)^3\delta^{(3)}(k_1+k_2+k_3)\sin\left(\frac{k_1\times k_2}{2}\right) \\
&\quad\times\frac{(k_1\times k_2)(\omega_1^2l_2^2+\omega_2^2l_1^2-2\omega_1\omega_2 l_1\cdot l_2)}{k_1^2k_2^2k_3^2}.
\end{aligned} \tag{3.38}$$

This precisely cancels (3.34) so that the sum of all the contributions to the three-point function $\langle W(k_1)W(k_2)W(k_3)\rangle$ at $O(g^4)$ vanishes. Thus

$$\langle W(k_1)W(k_2)W(k_3)\rangle=(2\pi)^9\delta^{(3)}(k_1)\delta^{(3)}(k_2)\delta^{(3)}(k_3)+O(g^5). \tag{3.39}$$

This is again consistent with the equivalence between noncommutative and commutative Chern-Simons theories.

4 Conclusions and discussion

We have calculated the two-point functions $\langle W(k)W(k')\rangle$ and $\langle W(k)O_{\mu\nu}(k')\rangle$, and the three-point function $\langle W(k_1)W(k_2)W(k_3)\rangle$ in noncommutative Chern-Simons theory, and compared them with their commutative counterparts. We found the equivalence between commutative and noncommutative Chern-Simons theories with respect to these observables persists at the first nontrivial order in perturbation theory.

The agreement in the two-point functions may seem more or less trivial since the topological nature of the theory strongly constrains the possible form of the correlation functions. In practice, however, we need to choose a gauge, which inevitably introduces metric dependence, and introduce a regulator to make the computation well-defined. We have acquired insight into interesting quantum aspects of the Seiberg-Witten map from the computation. First, the relation between the regularizations of the integral over σ and the commutator, which is closely connected with the Bianchi identity as we discussed in Subsection 2.2, did play an important role in the cancellations we found in the calculation of $\langle W(k)O_{\mu\nu}(k')\rangle$ in Subsection 3.3. Another interesting aspect we have encountered in the calculation is the quantum correction to the Seiberg-Witten map (3.25). These seem to provide us with some insight into how we should define the composite operators $W(k)$ and $O_{\mu\nu}(k)$ at the quantum level.

The agreement in the three-point function is more nontrivial. Although there is no dependence on the metric, $W(k)$ depends on $\theta^{\mu\nu}$ as well as its momentum so that the three-point function could be a nontrivial function of $k_1 \times k_2$. The nontrivial dependence on $k_1 \times k_2$ is not excluded by the topological nature of the theory, while the equivalence to the commutative theory requires it to vanish. We did find that it vanishes at the first nontrivial order in g.[9]

It would be an interesting question as to whether or not the equivalence between the commutative and noncommutative Chern-Simons theories persists to higher orders in g [32], or even nonperturbatively.[10] It has been noted in [33, 34, 35] that the level

[9]It was argued in [30, 31] that noncommutative Chern-Simons theory is a free theory from an analysis in the axial gauge. We would like to comment on the relation between this work and ours. First of all, correlation functions of composite operators, such as $W(k)$ or $O_{\mu\nu}(k)$ in our case, are in general nontrivial even in a free theory. For example, a vacuum expectation value of a Wilson loop is nontrivial in the free Abelian F^2 gauge theory. Therefore, our results are not immediate consequences of the observations in [30, 31]. Technically, however, our calculations could have been much simplified in the axial gauge. Although it was argued in [30, 31] that the axial gauge can be safely taken in noncommutative Chern-Simons theory, calculations involving open Wilson lines can be subtle because it is essential that we are able to perform integration by parts in proving various properties of open Wilson lines, such as the Bianchi identity, while the propagator does not decay at infinity in the axial gauge. For example, $W(k)$ with $k_2 = 0$ becomes trivial in the gauge $A_2 = 0$, but it is inconsistent with our result in the covariant gauge where $\langle W(k)O_{0i}(k')\rangle$ is nonvanishing.

[10]Rigorously speaking, pure noncommutative Chern-Simons theory without any additional degrees of freedom has not been realized in string theory. The realizations given in [18, 19] contain additional

of noncommutative Chern-Simons theory is quantized even for the $U(1)$ gauge group, while that for the commutative theory is not because of the difference in the gauge group topologies of the two cases [36]. This raises a question on the equivalence of the two theories at the nonperturbative level.[11] Clearly our perturbative computation does not address this issue, leaving this as an interesting future problem. If the equivalence holds nonperturbatively under the Seiberg-Witten map (2.1) up to possible quantum corrections to the map itself, correlation functions of $W(k)$ and $O_{\mu\nu}(k)$ are rather trivial in the sense that they are exactly given by their commutative counterparts. It would be an interesting future problem to construct more nontrivial observables, if any, in noncommutative Chern-Simons theory in this case.

Finally, let us discuss the generalization of our results to the $U(N)$ case. The map (2.1) between gauge-invariant observables on the commutative and noncommutative sides can be easily generalized to the $U(N)$ case by studying the coupling of multiple D-branes to the Ramond-Ramond potentials with a constant B field. The map in the case of $U(N)$ is simply given by taking the trace of (2.1):

$$\begin{aligned} \mathrm{tr}\ f_{12}(k) &= -\frac{1}{g\theta^{12}}\mathrm{tr}\left[W(k) - (2\pi)^3\delta^{(3)}(k)\mathbf{1}\right], \\ \mathrm{tr}\ f_{0i}(k) &= \mathrm{tr}\ O_{0i}(k) \quad \text{for} \quad i = 1, 2, \end{aligned} \tag{4.1}$$

where $\mathbf{1}$ is the identity matrix. A Feynman diagram of correlation functions involving tr $W(k)$ and tr $O_{\mu\nu}(k)$ can be evaluated by multiplying the contribution from the same diagram in the $U(1)$ case by an appropriate power of N.

All five diagrams of $\langle W(k)O_{\mu\nu}(k')\rangle$ at $O(g^3)$ not containing an internal loop scale as N^2 so that the cancellations we found remain intact. The two diagrams displayed in Figure 8 for $\langle W(k_1)W(k_2)W(k_3)\rangle$ at $O(g^4)$ scale as N. Therefore, the three-point function of tr $W(k)$ also vanishes at $O(g^4)$ in the $U(N)$ case. As for the one-loop corrections to the propagator, the cancellation of the nonplanar pieces persists and only the planar pieces produce a nonvanishing contribution just as in the case of $U(1)$. We should note that one-loop corrections to the propagator also exist on the commutative side in the case of $U(N)$ [27, 28, 29]. If we use the same regularization on the commutative and noncommutative sides, we have the same one-loop corrections on both sides so that one-loop corrections to the map (4.1) are absent in the $U(N)$ case. We thus conclude that the equivalence between $U(N)$ noncommutative and commutative Chern-Simons theories also holds at the first nontrivial order in g under the map (4.1) between gauge-invariant observables.

degrees of freedom which are important for quasi-particle or quasi-hole excitations of quantum Hall fluid, or to realize quantum Hall fluid with a boundary. Therefore, the equivalence between the commutative and noncommutative theories is not guaranteed by its embedding in string theory.

[11] We would like to thank D. Bak, M. M. Sheikh-Jabbari, and A. P. Polychronakos for drawing our attention to this point.

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Appendix A. Conventions and Feynman rules

The action of noncommutative Chern-Simons theory in terms of a canonically normalized gauge field is given by

$$S_{NCCS} = \frac{1}{2}\int d^3x\ \epsilon^{\mu\rho\nu}\left[A_\mu * \partial_\rho A_\nu - \frac{2ig}{3}A_\mu * A_\rho * A_\nu\right]. \tag{A.1}$$

Our Fourier transform convention is

$$A_\mu(x) = \int \frac{d^3p}{(2\pi)^3}\ e^{-ikx}A_\mu(k). \tag{A.2}$$

We use the standard covariant gauge-fixing term proportional to $(\partial\cdot A)^2$ and then take the Landau gauge. The propagator is given by

$$\langle A_\mu(p)A_\nu(q)\rangle = (2\pi)^3\delta^{(3)}(p+q)\epsilon_{\mu\rho\nu}\frac{p^\rho}{p^2}. \tag{A.3}$$

The three-point function of noncommutative gauge fields contracted with a cubic vertex is given by

$$\begin{aligned}
&\langle A_{\mu_1}(q_1)A_{\mu_2}(q_2)A_{\mu_3}(q_3)\rangle \\
&= -2ig(2\pi)^3\delta^{(3)}(q_1+q_2+q_3)\epsilon^{\nu_1\nu_2\nu_3} \\
&\quad\times\sin\left(\frac{q_1\times q_2}{2}\right)\frac{\epsilon_{\mu_1\rho_1\nu_1}q_1^{\rho_1}}{q_1^2}\frac{\epsilon_{\mu_2\rho_2\nu_2}q_2^{\rho_2}}{q_2^2}\frac{\epsilon_{\mu_3\rho_3\nu_3}q_3^{\rho_3}}{q_3^2}.
\end{aligned} \tag{A.4}$$

The following identities are useful:

$$\begin{aligned}
\epsilon_{\mu_1\rho_1\nu_1}\epsilon_{\mu_2\rho_2\nu_2} &= \delta_{\mu_1\mu_2}\delta_{\rho_1\rho_2}\delta_{\nu_1\nu_2} + \delta_{\mu_1\rho_2}\delta_{\rho_1\nu_2}\delta_{\nu_1\mu_2} + \delta_{\mu_1\nu_2}\delta_{\rho_1\mu_2}\delta_{\nu_1\rho_2} \\
&\quad -\delta_{\mu_1\mu_2}\delta_{\rho_1\nu_2}\delta_{\nu_1\rho_2} - \delta_{\mu_1\nu_2}\delta_{\rho_1\rho_2}\delta_{\nu_1\mu_2} - \delta_{\mu_1\rho_2}\delta_{\rho_1\mu_2}\delta_{\nu_1\nu_2}.
\end{aligned} \tag{A.5}$$

$$\begin{aligned}
&\epsilon_{\mu_1\rho_1\nu_1}\epsilon_{\mu_2\rho_2\nu_2}\epsilon_{\mu_3\rho_3\nu_3}\epsilon^{\nu_1\nu_2\nu_3} \\
&= \delta_{\mu_1\rho_2}\delta_{\rho_1\rho_3}\delta_{\mu_3\mu_2} + \delta_{\mu_1\rho_3}\delta_{\rho_1\mu_2}\delta_{\mu_3\rho_2} + \delta_{\mu_1\mu_3}\delta_{\rho_1\rho_2}\delta_{\rho_3\mu_2} + \delta_{\mu_1\mu_2}\delta_{\rho_1\mu_3}\delta_{\rho_3\rho_2} \\
&\quad -\delta_{\mu_1\mu_2}\delta_{\rho_1\rho_3}\delta_{\mu_3\rho_2} - \delta_{\mu_1\rho_3}\delta_{\rho_1\rho_2}\delta_{\mu_3\mu_2} - \delta_{\mu_1\rho_2}\delta_{\rho_1\mu_3}\delta_{\rho_3\mu_2} - \delta_{\mu_1\mu_3}\delta_{\rho_1\mu_2}\delta_{\rho_3\rho_2}.
\end{aligned} \tag{A.6}$$

Appendix B. $\langle W(k)O_{\mu\nu}(k')\rangle$

We present some details of the computations of $\langle W(k)O_{\mu\nu}(k')\rangle$ given as (3.10) through (3.14) in Subsection 3.3.

Diagram 1

The contribution to $\langle W(k)O_{\mu\nu}(k')\rangle$ from this diagram is given by

$$\begin{aligned}
&\langle W(k)O_{\mu\nu}(k')\rangle_{\text{Diagram 1}} \\
&= \frac{(ig)^2}{2}\int_\epsilon^{1-\epsilon} d\sigma \int \frac{d^3p}{(2\pi)^3}\; e^{-ik\times p\sigma+\frac{i}{2}k\times p} \\
&\qquad \times \left\langle l^{\mu_1}A_{\mu_1}(p)l^{\mu_2}A_{\mu_2}(k-p)(-ik'_\mu)A_\nu(k')\right\rangle - (\mu\leftrightarrow\nu).
\end{aligned} \tag{B.1}$$

This can be easily evaluated using (A.4) and (A.6) as follows:

$$\begin{aligned}
&\langle W(k)O_{\mu\nu}(k')\rangle_{\text{Diagram 1}} \\
&= -\frac{(ig)^3}{2}(2\pi)^3\delta^{(3)}(k+k')\int_\epsilon^{1-\epsilon} d\sigma \int \frac{d^3p}{(2\pi)^3}\;\frac{e^{-ik\times p\sigma}-e^{ik\times p(1-\sigma)}}{p^2(k-p)^2} \\
&\qquad \times \left\{(k\times p)(k_\mu l_\nu - k_\nu l_\mu) - l^2(k_\mu p_\nu - k_\nu p_\mu)\right\},
\end{aligned} \tag{B.2}$$

using $l\cdot p = k\times p$. The part involving $e^{ik\times p(1-\sigma)}$ can be transformed to the part involving $e^{ik\times p\sigma}$ by the change of variables $\sigma' = 1-\sigma$ and $p' = k-p$. Therefore, we have

$$\begin{aligned}
&\langle W(k)O_{\mu\nu}(k')\rangle_{\text{Diagram 1}} \\
&= -(ig)^3(2\pi)^3\delta^{(3)}(k+k')\int_\epsilon^{1-\epsilon} d\sigma \int \frac{d^3p}{(2\pi)^3}\;\frac{e^{-ik\times p\sigma}}{p^2(k-p)^2} \\
&\qquad \times \left\{(k\times p)(k_\mu l_\nu - k_\nu l_\mu) - l^2(k_\mu p_\nu - k_\nu p_\mu)\right\}.
\end{aligned} \tag{B.3}$$

Diagram 2

The contribution to $\langle W(k)O_{\mu\nu}(k')\rangle$ from this diagram is given by

$$\begin{aligned}
&\langle W(k)O_{\mu\nu}(k')\rangle_{\text{Diagram 2}} \\
&= \frac{(ig)^2}{2}\int_\epsilon^{1-\epsilon} d\sigma \int \frac{d^3p}{(2\pi)^3}\; e^{-ik\times p\sigma+\frac{i}{2}k\times p}\;(-ig)\int \frac{d^3q}{(2\pi)^3}\; e^{-ik'\times q\sigma+\frac{i}{2}k'\times q} \\
&\qquad \times \langle l^{\mu_1}A_{\mu_1}(p)l^{\mu_2}A_{\mu_2}(k-p)A_\mu(q)A_\nu(k'-q)\rangle - (\mu\leftrightarrow\nu).
\end{aligned} \tag{B.4}$$

There are two nonvanishing ways to contract the four gauge fields. Using the propagator (A.3), it is evaluated as

$$\begin{aligned}
&\langle W(k)O_{\mu\nu}(k')\rangle_{\text{Diagram 2}} \\
&= -\frac{(ig)^3}{2}(2\pi)^3\delta^{(3)}(k+k')\int_\epsilon^{1-\epsilon} d\sigma \int \frac{d^3p}{(2\pi)^3}\left\{e^{ik\times p(1-\sigma-\epsilon)} - e^{-ik\times p(\sigma-\epsilon)}\right\} \\
&\qquad \times l^{\mu_1}\epsilon_{\mu_1\rho_1\mu}l^{\mu_2}\epsilon_{\mu_2\rho_2\nu}\frac{p^{\rho_1}(k-p)^{\rho_2}}{p^2(k-p)^2} - (\mu\leftrightarrow\nu).
\end{aligned} \tag{B.5}$$

Using the identity (A.5), this can be further evaluated as

$$
\begin{aligned}
&\langle W(k)O_{\mu\nu}(k')\rangle_{\text{Diagram 2}}\\
&= -\frac{(ig)^3}{2}(2\pi)^3\delta^{(3)}(k+k')\int_{\epsilon}^{1-\epsilon}d\sigma\int\frac{d^3p}{(2\pi)^3}\frac{e^{ik\times p(1-\sigma-\epsilon)}-e^{-ik\times p(\sigma-\epsilon)}}{p^2(k-p)^2}\\
&\qquad\times\Big\{(k\times p)(k_\mu l_\nu - k_\nu l_\mu) - l^2(k_\mu p_\nu - k_\nu p_\mu)\Big\}\\
&= (ig)^3(2\pi)^3\delta^{(3)}(k+k')\int_{\epsilon}^{1-\epsilon}d\sigma\int\frac{d^3p}{(2\pi)^3}\frac{e^{-ik\times p(\sigma-\epsilon)}}{p^2(k-p)^2}\\
&\qquad\times\Big\{(k\times p)(k_\mu l_\nu - k_\nu l_\mu) - l^2(k_\mu p_\nu - k_\nu p_\mu)\Big\}\\
&= (ig)^3(2\pi)^3\delta^{(3)}(k+k')\int_{0}^{1-2\epsilon}d\sigma\int\frac{d^3p}{(2\pi)^3}\frac{e^{-ik\times p\sigma}}{p^2(k-p)^2}\\
&\qquad\times\Big\{(k\times p)(k_\mu l_\nu - k_\nu l_\mu) - l^2(k_\mu p_\nu - k_\nu p_\mu)\Big\}\,,
\end{aligned}
\tag{B.6}
$$

where we have used the same change of variables as before.

Diagram 3

The contribution to $\langle W(k)O_{\mu\nu}(k')\rangle$ from this diagram is given by

$$
\begin{aligned}
&\langle W(k)O_{\mu\nu}(k')\rangle_{\text{Diagram 3}}\\
&= (ig)^2\int_{\epsilon}^{1-\epsilon}d\sigma\int\frac{d^3p}{(2\pi)^3}\;e^{-ik'\times p\sigma+\frac{i}{2}k'\times p}\\
&\qquad\times\langle l^{\mu_1}A_{\mu_1}(k)l'^{\mu_2}A_{\mu_2}(p)(-i)(k'-p)_\mu A_\nu(k'-p)\rangle - (\mu\leftrightarrow\nu),
\end{aligned}
\tag{B.7}
$$

where $l'^{\mu} = (k'\theta)^\mu = k'_\nu\theta^{\nu\mu}$. As in the case of Diagram 1, this can be evaluated using (A.4) and (A.6) as follows:

$$
\begin{aligned}
&\langle W(k)O_{\mu\nu}(k')\rangle_{\text{Diagram 3}}\\
&= (ig)^3(2\pi)^3\delta^{(3)}(k+k')\int_{\epsilon}^{1-\epsilon}d\sigma\int\frac{d^3p}{(2\pi)^3}\;\Big\{e^{ik\times p\sigma} - e^{-ik\times p(1-\sigma)}\Big\}\\
&\quad\times\left[\frac{(k\times p)\{(-k^2-k\cdot p)(k+p)_\mu l_\nu + (k\times p)p_\mu k_\nu\}}{k^2p^2(k+p)^2} + \frac{l^2k_\mu p_\nu}{k^2p^2}\right]\\
&\qquad - (\mu\leftrightarrow\nu)\\
&= (ig)^3(2\pi)^3\delta^{(3)}(k+k')\int_{\epsilon}^{1-\epsilon}d\sigma\int\frac{d^3p}{(2\pi)^3}\;e^{ik\times p\sigma}\\
&\quad\times\left[\frac{k\times p}{k^2p^2(k+p)^2}\Big\{-(k^2+k\cdot p)(k_\mu l_\nu - k_\nu l_\mu) - (k^2+2k\cdot p)(p_\mu l_\nu - p_\nu l_\mu)\right.\\
&\qquad\left. + 2(k\times p)(p_\mu k_\nu - p_\nu k_\mu)\Big\} + \frac{2l^2(k_\mu p_\nu - k_\nu p_\mu)}{k^2p^2}\right],
\end{aligned}
\tag{B.8}
$$

where we have used the appropriate change of variables: either $\sigma' = 1-\sigma$, $p' = -k-p$ or $\sigma' = 1-\sigma$, $p' = -p$. Since

$$(k \times p)\, e^{ik\times p\sigma} = -i\frac{\partial}{\partial\sigma}\,(e^{ik\times p\sigma}), \tag{B.9}$$

only surface terms of the σ integral contribute for the terms proportional to $k\times p$. We separate the bulk terms and the surface terms as follows:

$$\begin{aligned}
&\langle W(k)O_{\mu\nu}(k')\rangle_{\text{Diagram 3}} \\
&= 2(ig)^3(2\pi)^3\delta^{(3)}(k+k')\int_\epsilon^{1-\epsilon} d\sigma \int \frac{d^3p}{(2\pi)^3}\, e^{ik\times p\sigma}\, \frac{l^2(k_\mu p_\nu - k_\nu p_\mu)}{k^2p^2} \\
&\quad -i(ig)^3(2\pi)^3\delta^{(3)}(k+k')\int \frac{d^3p}{(2\pi)^3}\, \frac{e^{ik\times p(1-\epsilon)} - e^{ik\times p\epsilon}}{k^2p^2(k+p)^2} \\
&\quad\quad \times\Big[-(k^2+k\cdot p)(k_\mu l_\nu - k_\nu l_\mu) - (k^2+2k\cdot p)(p_\mu l_\nu - p_\nu l_\mu) \\
&\quad\quad\quad + 2(k\times p)(p_\mu k_\nu - p_\nu k_\mu)\Big].
\end{aligned} \tag{B.10}$$

Diagram 4

In order to evaluate this diagram, the following piece of $O_{\mu\nu}(k)$ at order g^2 is necessary:

$$-(ig)^2\int d^3x \int_{2\epsilon}^{1-\epsilon} d\sigma\,\Big[l\cdot A(x+l\sigma) * A_\mu(x+l\epsilon) * A_\nu(x) * e^{ikx}\Big] - (\mu \leftrightarrow \nu). \tag{B.11}$$

In momentum space, it is given by

$$\begin{aligned}
&-(ig)^2\int_{2\epsilon}^{1-\epsilon} d\sigma \int \frac{d^3p_1}{(2\pi)^3}\frac{d^3p_2}{(2\pi)^3}\frac{d^3p_3}{(2\pi)^3}(2\pi)^3\delta^{(3)}(k-p_1-p_2-p_3) \\
&\quad\times e^{-\frac{i}{2}(p_1\times p_2 + p_1\times p_3 + p_2\times p_3) - ik\times p_1\sigma - ik\times p_2\epsilon}\; l\cdot A(p_1)A_\mu(p_2)A_\nu(p_3) - (\mu\leftrightarrow\nu).
\end{aligned} \tag{B.12}$$

We contract the gauge field coming from the open Wilson line with one of the gauge fields in the commutator to give

$$\begin{aligned}
&-(ig)^2\int_{2\epsilon}^{1-\epsilon} d\sigma\int\frac{d^3p}{(2\pi)^3}\left\{e^{-ik\times p(\sigma-\epsilon)} - e^{ik\times p(1-\sigma)}\right\}\frac{l^\lambda p^\rho}{p^2}\left\{\epsilon_{\lambda\rho\mu}A_\nu(k) - \epsilon_{\lambda\rho\nu}A_\mu(k)\right\} \\
&= -2(ig)^2\int_{2\epsilon}^{1-\epsilon} d\sigma\int\frac{d^3p}{(2\pi)^3}\; e^{-ik\times p(\sigma-\epsilon)}\;\frac{l^\lambda p^\rho}{p^2}\left\{\epsilon_{\lambda\rho\mu}A_\nu(k) - \epsilon_{\lambda\rho\nu}A_\mu(k)\right\}.
\end{aligned} \tag{B.13}$$

The contribution to $\langle W(k)O_{\mu\nu}(k')\rangle$ from this diagram is given by contracting the remaining gauge field in the commutator with the gauge field from the other open Wilson line:

$$
\begin{aligned}
&\langle W(k)O_{\mu\nu}(k')\rangle_{\text{Diagram 4}} \\
&= -2(ig)^3 \int_{2\epsilon}^{1-\epsilon} d\sigma \int \frac{d^3p}{(2\pi)^3}\ e^{-ik'\times p(\sigma-\epsilon)}\ \frac{l'^{\lambda}p^{\rho}}{p^2} \\
&\quad \times \left\{ l^{\mu_1}\epsilon_{\lambda\rho\mu} \left\langle A_{\mu_1}(k)A_{\nu}(k')\right\rangle - l^{\mu_1}\epsilon_{\lambda\rho\nu} \left\langle A_{\mu_1}(k)A_{\mu}(k')\right\rangle \right\}, \qquad \text{(B.14)}
\end{aligned}
$$

where $l'^{\mu} = (k'\theta)^{\mu} = k'_{\nu}\theta^{\nu\mu}$ as before. This can be evaluated using (A.3) and (A.5):

$$
\begin{aligned}
&\langle W(k)O_{\mu\nu}(k')\rangle_{\text{Diagram 4}} \\
&= 2(ig)^3(2\pi)^3\delta^{(3)}(k+k') \int_{2\epsilon}^{1-\epsilon} d\sigma \int \frac{d^3p}{(2\pi)^3}\ \frac{e^{ik\times p(\sigma-\epsilon)}}{k^2p^2} \\
&\quad \times \Big\{ (k\times p)(k_{\mu}l_{\nu} - k_{\nu}l_{\mu}) - l^2(k_{\mu}p_{\nu} - k_{\nu}p_{\mu}) \Big\} \\
&= -2(ig)^3(2\pi)^3\delta^{(3)}(k+k') \int_{\epsilon}^{1-2\epsilon} d\sigma \int \frac{d^3p}{(2\pi)^3}\ e^{ik\times p\sigma}\ \frac{l^2(k_{\mu}p_{\nu} - k_{\nu}p_{\mu})}{k^2p^2} \\
&\quad -2i(ig)^3(2\pi)^3\delta^{(3)}(k+k') \int \frac{d^3p}{(2\pi)^3}\ \Big\{ e^{ik\times p(1-2\epsilon)} - e^{ik\times p\epsilon} \Big\} \frac{k_{\mu}l_{\nu} - k_{\nu}l_{\mu}}{k^2p^2}, \qquad \text{(B.15)}
\end{aligned}
$$

where we have separated the bulk contribution and the surface contribution as before.

Diagram 5

The contribution to $\langle W(k)O_{\mu\nu}(k')\rangle$ from this diagram is given by

$$
\begin{aligned}
&\langle W(k)O_{\mu\nu}(k')\rangle_{\text{Diagram 5}} \\
&= -(ig)^2 \int \frac{d^3p}{(2\pi)^3}\ e^{-ik'\times p\epsilon + \frac{i}{2}k'\times p} \left\langle l^{\mu_1} A_{\mu_1}(k)A_{\mu}(p)A_{\nu}(k'-p)\right\rangle - (\mu \leftrightarrow \nu). \qquad \text{(B.16)}
\end{aligned}
$$

Using (A.4) and (A.6), this can be evaluated as follows:

$$
\begin{aligned}
&\langle W(k)O_{\mu\nu}(k')\rangle_{\text{Diagram 5}} \\
&= -i(ig)^3(2\pi)^3\delta^{(3)}(k+k') \int \frac{d^3p}{(2\pi)^3}\ \frac{e^{ik\times p\epsilon} - e^{-ik\times p(1-\epsilon)}}{k^2p^2(k+p)^2} \\
&\quad \times \Big[-2(k\times p)(k_{\mu}p_{\nu} - k_{\nu}p_{\mu}) - (2p^2 + k\cdot p)(l_{\mu}k_{\nu} - l_{\nu}k_{\mu}) \\
&\qquad + (k^2 + 2k\cdot p)(l_{\mu}p_{\nu} - l_{\nu}p_{\mu}) \Big]. \qquad \text{(B.17)}
\end{aligned}
$$

It may not be manifest, but the integrand excluding $e^{ik\times p\epsilon} - e^{-ik\times p(1-\epsilon)}$ is invariant under the change of variables $p' = -k - p$. This is manifest in the denominator. For the numerator, we can verify the following identity:

$$
\begin{aligned}
&-2(k\times p)(k_\mu p_\nu - k_\nu p_\mu) - (2p^2 + k\cdot p)(l_\mu k_\nu - l_\nu k_\mu) + (k^2 + 2k\cdot p)(l_\mu p_\nu - l_\nu p_\mu) \\
&= -2(k\times p')(k_\mu p'_\nu - k_\nu p'_\mu) - (2p'^2 + k\cdot p')(l_\mu k_\nu - l_\nu k_\mu) \\
&\qquad + (k^2 + 2k\cdot p')(l_\mu p'_\nu - l_\nu p'_\mu).
\end{aligned}
\tag{B.18}
$$

Therefore, we have

$$
\begin{aligned}
&\langle W(k) O_{\mu\nu}(k')\rangle_{\text{Diagram 5}} \\
&= i(ig)^3 (2\pi)^3 \delta^{(3)}(k+k') \int \frac{d^3p}{(2\pi)^3} \, \frac{e^{ik\times p(1-\epsilon)} - e^{ik\times p\epsilon}}{k^2 p^2 (k+p)^2} \\
&\quad \times \Big[-2(k\times p)(k_\mu p_\nu - k_\nu p_\mu) - (2p^2 + k\cdot p)(l_\mu k_\nu - l_\nu k_\mu) \\
&\qquad\qquad + (k^2 + 2k\cdot p)(l_\mu p_\nu - l_\nu p_\mu)\Big].
\end{aligned}
\tag{B.19}
$$

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